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Fluxons in a system of two Josephson junctions

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Abstract. The non-linear system of equations of the sine-Gordon type, describing waves in Josephson superlattices, is investigated in the framework of the piecewise-linear model. A detailed quantitative analysis of the fluxon states and their bifurcations in the simple case of the system of two identical semi-infinite Josephson junctions, placed in an external magnetic field, is carried out.

1. Introduction

The possibilities of using solitons in Josephson junctions (JJ) in memory and switching elements of computers have been repeatedly discussed and are well known [1]. But the concrete realisation of their advantages requires a detailed theoretical and experimental study of physical phenomena in JJ. In particular, for applications it is of interest to know the character of the interaction of fluxons with the boundaries of real physical systems and with inhomogeneities in the internal structure of JJ systems (JJS). Usually this interaction has been studied by means of different variants of perturbation theory [2], in which the moving one- or multi-soliton states in a homogeneous infinite junction are taken as unperturbed states. The effect of the perturbation in this approach is taken into account by considering the soliton as a deformable particle which is acted upon by friction forces, causing the dissipation of the energy in the junction. But such a description becomes inapplicable when solitons are strongly deformed by the interaction with the inhomogeneity or with the boundary of the physical system. This happens, for example, when the fluxon is localised on the inhomogeneity or the fluxon tears away from the boundary of the JJ in a strong magnetic field. Analytical and numerical methods which in principle allow one to obtain exact solutions for problems of this type were considered in [3-5].

But it should be emphasised that a detailed analysis of the analytical properties of the exact solution does not always appear to be possible. This turns out to be the case when studying some properties of the JJ (e.g. the stability of solutions, the dependence of the supercurrent on external magnetic field, etc.). It is known that the phase difference between the macroscopic quantum wavefunctions of two superconductors, forming the junction, possesses direct physical meaning and is described by the sine-Gordon equation. The solutions of this equation can be expressed, in principle, in terms of elliptic functions [3, 5], whose properties essentially depend on certain parameters. However, when one attempts to formulate the boundary conditions, which are of physical interest, there appear complicated transcendent equations for these parameters. To deal with this difficulty, Gal'pern and Filippov [5] proposed the piecewise-linear model, which admits a solution in terms of elementary functions and is in good agreement with numerical calculations of the exact expressions. It was shown that fluxon bound states on attractive inhomogeneities in the JJ exhibit non-trivial bifurcations (i.e. appearance or disappearance of a number of solutions for some critical values of the external parameters) under a change of the external magnetic field. We have used this model for the detailed quantitative study of the bound states of fluxons in JJ and their bifurcations [6, 7].

A certain amount of attention has recently been given to Josephson superlattices, i.e. systems consisting of alternating layers S-N-S-... or S-I-S-... [8, 9]. This opens up new possibilities for studying the radiation from multilayer systems [10] and fluxon interaction, since by varying parameters of the JJS it is possible to change the fluxon's characteristics. Some of these parameters were mentioned above; we also note the influence which geometrical scales and the coupling between different junctions, forming the JJS, have on the properties of the whole system. For example, the system with two different weakly coupled homogeneous junctions, placed in a constant magnetic field, was considered in [11]. In this case a vortex lattice arises in one of the junctions, while the fluxon in the other junction will be moving in the field of the periodic potential created by this vortex lattice. The effect of weak coupling causes the appearance of an additional resonance peak on the I-V characteristic, which is absent on the I-V characteristic of an isolated JJ. Besides, in [12] it was shown that if the velocities of fluxons moving in two neighbouring junctions differ by a small amount, then their mutual capture occurs. We also note the interesting results of the investigation of different dynamical processes with fluxons in JJS presented in [13]. In particular, the destructive collision between a low-frequency breather and a fast fluxon, belonging to the mate junction, was considered in the framework of the adiabatic approximation (i.e. without taking into account the radiative losses [2]) and a condition was found, under which a breather decays into a free fluxon-antifluxon pair. The radiative effects which accompany the interaction between fluxons from the same or different junctions were also considered.

In the present paper, with the help of the piecewise-linear model, we investigate the non-linear system of equations of the sine-Gordon type, which describes waves in the Josephson superlattice. We perform a detailed quantitative analysis of the fluxon states and their bifurcations in the simple case of a system of two identical semi-infinite JJ placed in an external magnetic field. The static distributions of the magnetic flux in an inhomogeneous Josephson system were also investigated. We have shown that in the inhomogeneous case a richer picture of the states is obtained and have studied in detail some of those states which are stable. Such analysis of the stable static states (and their bifurcations) is of interest precisely because they are the states of equilibrium through which the system can pass during its time evolution.

2. The piecewise-linear model for the case of JJS

Let us consider the system of two long† JJ of the S-I-S type, placed in an external magnetic field of intensity h_0 (see figure 1). Let us suppose that the coupling (interaction) between the junctions is weak. By this we mean the following. As is well known,

† A JJ is usually called long if the condition $w \ll \lambda_J \ll l$ is satisfied, where w is the width and l is the length of the JJ [2].

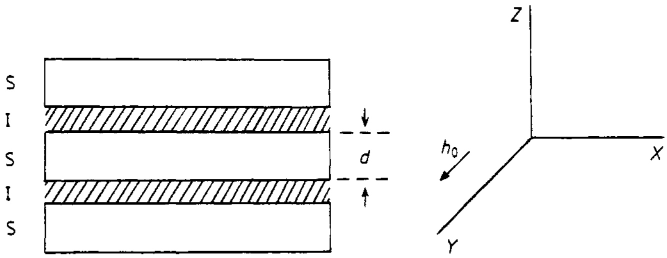

Figure 1.

the penetration region of electromagnetic waves into a JJ is restricted to the thin layer of dielectric plus the neighbouring regions of superconducting films with thickness of the order of λ_L. Thus the field in the I layer can interact with external fields on the edges of the junction only (assuming that the thickness d of the superconducting layers is much greater than the London penetration depth λ_L). Then, because of the very small thickness of the I layer, the coupling between the JJ is very small. At any point of the I layer of each JJ a magnetic field will be observed, created by the other junction, but decreased by the factor $\exp(-d/\lambda_L)$ owing to screening by the superconducting strip of the thickness d, which separates the I layers of the two JJ.

The penetration depth of the electric field is negligibly small compared with λ_L and therefore the dependence on the electric field is not discussed.

The derivation of a system of equations describing the wave in the system of two weakly coupled junctions is presented in sufficient detail in [12], while for the case of a system consisting of a JJ and a waveguide, formed by superconducting layers and filled by dielectric, is given in [11]. Here we shall make use of these relations [12], expressed in dimensionless variables (as regards dimensions see, e.g., [1, 2, 14]):

$$\begin{aligned} &\phi_1''(x,t)+\gamma_1\phi_2''(x,t)-\alpha_1\dot{\phi}_1(x,t)-\ddot{\phi}_1(x,t)=\sin\phi_1(x,t) \\ &\phi_2''(x,t)+\gamma_2\phi_1''(x,t)-\alpha_2\dot{\phi}_2(x,t)-\ddot{\phi}_2(x,t)=\sin\phi_2(x,t) \end{aligned} \tag{1}$$

where $\dot{\phi}_k(x,t)=\partial\phi_k(x,t)/\partial t$ and $\phi_k'(x,t)=\partial\phi(x,t)/\partial x$; $\phi_k(x,t)$ is the phase difference between the macroscopic quantum wavefunctions of the two superconductors in the kth JJ ($k=1,2$); $\phi_k(x,t)=2\pi\Phi_k(x,t)/\Phi_0$ where $\Phi_k(x,t)$ is the magnetic flux and $\Phi_0=\pi\hbar c/e$ is a quantum of the magnetic flux; α_k is the dissipation coefficient and γ_k is the parameter mentioned above, which characterises the coupling between the JJ of a system. Because of the identity of both junctions $\gamma_1=\gamma_2=\gamma$ and $\alpha_1=\alpha_2=\alpha$. The boundary condition for the problem, described by (1), has the form

$$\phi_1'(x,t)=h_1 \qquad \phi_2'(x,t)=h_2. \tag{2}$$

When investigating the system (1) we shall take into account only those solutions which are stable with respect to small fluctuations. To this end we represent perturbed solutions as

$$\phi_k(x,t)=\phi_k(x)+\psi_k(x)\exp[-(\tfrac{1}{2}\alpha+i\bar{\omega})t] \tag{3}$$

and, substituting (3) in (1), we obtain the equation for static states in JJS

$$\phi_{1(2)}''(x)+\gamma\phi_{2(1)}''(x)=\sin\phi_{1(2)}(x) \tag{4}$$

and the stability condition for these states

$$-[\psi_{1(2)}''(x)+\gamma\psi_{2(1)}''(x)]+\psi_{1(2)}(x)\cos\phi_{1(2)}(x)=\omega^2\psi_{1(2)}(x) \qquad \psi_{1(2)}'(0)=0 \tag{5}$$

where $\omega^2 = \frac{1}{4}\alpha^2 + \bar{\omega}^2$. Thus $\psi(x)$ are eigenfunctions of the linear boundary problem (5) with eigenvalues ω^2. From (3) it now follows that the perturbed solution will not be increasing (i.e. will be stable) provided that the lowest eigenvalue ω_0^2 is positive. The expansion (3) describes the evolution in time of any small perturbation [5, 15]. If $\omega_0^2 > 0$, then ω_0 defines the response frequency of the system $\phi(x)$ to any small perturbation, if $\omega_0^2 < 0$, then the state is unstable, but when $|\bar{\omega}_0^2| = |\omega_0^2| + \frac{1}{4}\alpha^2 \ll 1$ it is possible to speak about its lifetime. Thus, for the complete characterisation of the static state it is necessary to know the lowest eigenfrequency.

For the Josephson currents in the identical junctions under consideration we shall employ the continuously differentiable approximation [5-7]

$$1 - \cos\phi_k(x) \simeq \tfrac{1}{2}(-1)^{N_k}(\phi_k(x) - \pi N_k)^2 + \tfrac{1}{8}\pi^2[1 - (-1)^{N_k}] \tag{6}$$

on the intervals $(N_{1(2)} - \frac{1}{2})\pi \leq \phi_{1(2)}(x) \leq (N_{1(2)} + \frac{1}{2})\pi$, where $N_{1(2)}$ are integers. Then differentiating (6) with respect to ϕ we obtain for (4) the piecewise-linear approximation

$$\phi''_{1(2)}(x) + \gamma\phi''_{2(1)}(x) = (-1)^{N_{1(2)}}(\phi_{1(2)}(x) - \pi N_{1(2)}) \tag{7}$$

while for the boundary value problem (5) one more differentiation gives a piecewise-continuous approximation for $\cos\phi$:

$$-[\psi''_{1(2)}(x) + \gamma\psi''_{2(1)}(x)] + (-1)^{N_{1(2)}}\psi_{1(2)}(x) = \omega^2\psi_{1(2)}(x). \tag{8}$$

If one sets $\gamma = 0$ in (7) and (8), then one evidently obtains equations describing the static states of non-interacting JJ and their eigenfrequencies. In other words, the case of $\gamma = 0$ is simply reduced to the problem of isolated JJ in an external field (considered by us in [6, 7]). Of physical interest is a weakly coupled system, which by definition corresponds to the values $0 < \gamma \ll 1$. In this case an energy of the interaction between magnetic fields in different junctions is defined by

$$W = -\gamma \int \frac{\partial\phi_1}{\partial x}\frac{\partial\phi_2}{\partial x}\,dx.$$

Therefore such an interaction can be described by means of perturbation theory in the parameter γ.

Let us first consider the system (7). First we will find solutions of equations (7) in the simplest case of N_1 and N_2 being equal. When both N_1 and N_2 are even, for the function

$$U(x) = \phi_1(x) + \phi_2(x) \tag{9}$$

we obtain from (7) the equation

$$(1 + \gamma)U''(x) = U(x) - \pi N \tag{10a}$$

where $N = N_1 + N_2 = 2N_1$. The solution of (10a) on the intervals $x \in [\bar{x}_N, \bar{x}_{N+1}] = \bar{I}_N$, for which $U(x) \in I_N = [U(\bar{x}_N), U(\bar{x}_{N+1})]$ (where $U(\bar{x}_N) = N\pi$), has the form

$$U(x) - \pi N = a_N \sinh(\tau(x - \bar{x}_N)) = a'_N \sinh(\tau(x - \bar{x}_{N+1})) \tag{11a}$$

where $\tau = (1 + \gamma)^{-1/2}$, and a_N and a'_N are constants. They are easily expressed through the boundary values (2) (taking into account (9)) and the parameter x_N. Obtaining $\phi_2(x)$ from (9) and substituting it into the first equation of the system (7), we get

$$(1 - \gamma)\phi''_1(x) + \gamma U''(x) = \phi_1(x) - \pi N_1. \tag{12}$$

The solution of this equation has the form

$$\phi_1(x) = \tfrac{1}{2}\pi[b_{N_1} \sinh p(x - \bar{x}_{N_1}) - \cosh p(x - \bar{x}_{N_1})] + \tfrac{1}{2}U(x)$$
$$= \tfrac{1}{2}\pi[b'_{N_1} \sinh p(x - \bar{x}_{N_1+1}) + \cosh p(x - \bar{x}_{N_1+1})] + \tfrac{1}{2}U(x) \qquad (13a)$$

where the function $U(x)$ is explicitly given by (11a), $p = (1-\gamma)^{-1/2}$, and b_N and b'_N are constants.

In the case of odd N_1 and N_2, the system (7) yields the equation

$$(1+\gamma)U''(x) = -(U(x) - \pi N) \qquad (10b)$$

which has solutions of the form

$$U(x) - \pi N = a_N \sin(\tau(x - \bar{x}_N)) = a'_N \sin(\tau(x - \bar{x}_{N+1})). \qquad (11b)$$

Hence with the aid of (9) we find $\phi_1(x)$ for this case

$$\phi_1(x) = \tfrac{1}{2}\pi[b_{N_1} \sin \pi(x - \bar{x}_{N_1}) - \cos p(x - \bar{x}_{N_1})] + \tfrac{1}{2}U(x)$$
$$= \tfrac{1}{2}\pi[b'_{N_1} \sin p(x - \bar{x}_{N_1+1}) + \cos p(x - \bar{x}_{N_1+1})] + \tfrac{1}{2}U(x). \qquad (13b)$$

Let us now consider the more general case in which N_1 is not equal to N_2. If N_1 is even and N_2 is odd (or vice versa) the system (7) can be reduced to a differential equation of fourth order in $\phi_1(x)$ (or $\phi_2(x)$, respectively):

$$(1-\gamma^2)\phi_1''''(x) = \phi_1(x) - \pi N_1. \qquad (14)$$

Its solution on the interval $\bar{I}_{N_1}$ is presented in the form (valid for both cases)

$$\phi_1(x) - \pi N_1 = \tfrac{1}{2}\pi(a_{N_1} \sinh X_{N_1} - \cosh X_{N_1}) + b_{N_1} \sinh X_{N_1} + d_{N_1}(\cos X_{N_1} - \cosh X_{N_1})$$
$$= \tfrac{1}{2}\pi(a'_{N_1} \sinh X_{N_1+1} + \cosh X_{N_1+1}) + b'_{N_1} \sin X_{N_1+1}$$
$$+ d'_{N_1}(\cos X_{N_1+1} - \cosh X_{N_1+1}) \qquad (15)$$

where $X_{N_1} = \varkappa(x - \bar{x}_{N_1})$ and $X_{N_1+1} = \varkappa(x - \bar{x}_{N_1+1})$ with $\varkappa = (1-\gamma^2)^{-1/4}$.

Equating the solutions (13) or (15) at the points $\bar{x}_{N_1}$, it is possible to construct an expression for $\phi_1(x)$ which is valid for all values of the variable x. Considering the second equation of the system (7), it is also possible to obtain an analogous expression for $\phi_2(x)$ as well.

To define ω^2 and $\psi(x)$ we return to (8). As a matter of fact, the procedure for solving this system is exactly the same as for (7). We only note that it is more convenient to look for a solution not in the form of a function $\psi_1(x)$, but to first find $F_1(x) = \psi'_1(x)/\psi_1(x)$. Then

$$\left.\begin{aligned} F_1(x) &= \frac{qp \sinh(qpx + \theta) + \nu'_1(x)}{\cosh(qpx + \theta) + \nu_1(x)} \\ \nu_1(x) &= \theta_1 \cosh(q\tau x + \theta_2) \end{aligned}\right\} \text{for odd } N_1$$
$$\left.\begin{aligned} F_1(x) &= \frac{sp \sin(spx + \theta) + \nu'_1(x)}{\nu_1(x) - \cos(spx + \theta)} \\ \nu_1(x) &= \theta_1 \cos(s\tau x + \theta_2) \end{aligned}\right\} \text{for even } N_1 \qquad (16)$$

where p and τ have the same meaning as above, $q^2=1-\omega_0^2$ and $s^2=1+\omega_0^2$. On the boundaries $F_1(x_0)=0$. Using the solutions (11)-(16), it is possible, for given $\bar{x}_N$, to eliminate the unknown parameters and to obtain the equation for ω^2. Analogous expressions are also obtainable for $\psi_2(x)$ (or for $F_2(x)$). Thus the formulae (13)-(16), together with boundary conditions and requirement of continuity, allow one to find bound states and calculate their eigenfrequencies. If an eigenvalue of this problem vanishes (i.e. $\omega^2=0$), then a bifurcation can take place. Therefore the equation

$$\omega^2=0 \tag{17}$$

defines a surface of bifurcations for the problem under consideration (as we shall see later, bifurcations do take place). We note that equation (17) explicitly contains parameters of the problem (magnetic field intensity h_0, weak coupling constant γ and so on). Let us consider some examples.

3. A system of homogeneous junctions

A system of homogeneous semi-infinite identical JJ may serve as a simple example. In this case the boundary conditions (2) will take the form

$$\phi'(0)=h_0 \qquad \phi'(\infty)=0. \tag{18}$$

With the help of relations (11)-(15) we can construct the states of magnetic flux in each junction. For definiteness let us take the case when $\phi_1(0)\in I_1$ and $\phi_1(\infty)\in I_2$, while N_2 is an odd number. Then

$$\phi_1(x)=\begin{cases}\tfrac{1}{2}\pi[\cos p(x-\bar{x}_2)-\tan(p\bar{x}_2)\sin p(x-\bar{x}_2)] & \\ +(h_0/\tau)\sin\tau(x-\bar{x}_2)+\pi & 0<x<\bar{x}_2 \\ 2\pi-\tfrac{1}{2}\pi\exp[-p(x-\bar{x}_2)] & \bar{x}_2<x.\end{cases} \tag{19}$$

From the continuity condition for the derivative of $\phi_1(x)$ at the point $\bar{x}_2$ we obtain the following equation for the unknown parameter $\bar{x}_2$:

$$h_0=\tfrac{1}{2}\pi p\cos(\tau\bar{x}_2)[1+\tan(p\bar{x}_2)]. \tag{20}$$

The bifurcation points $\bar{x}_2=\bar{x}_{2c}$ are defined by the condition $\mathrm{d}h_0/\mathrm{d}\bar{x}_2=0$, i.e.

$$\tau\tan(\tau\bar{x}_2)[(1/\sqrt{2})+\cos(2p\bar{x}_2+\tfrac{1}{4}\pi)]=\sqrt{2}p. \tag{21}$$

The substitution of the above transcendental equation into (20) leads to a bifurcation curve $h_0(\gamma)$. Thus the qualitative structure of static states is the following. For given boundary values of the magnetic field there usually exist several states, differing in the amount and distribution of magnetic flux. Some of these states turn out to be locally stable and as seen from the bifurcation curve on figure 2, for given values of boundary fields there exist more than one stable state. This means that the system under consideration is very sensitive to the variation of the parameter γ.

We will now construct and investigate the stability of the other static state $\phi_1(x)$, which is still located on the same intervals, but the corresponding values of N_2 are even. For this state, from the continuity requirement for $\mathrm{d}\phi_1(x)/\mathrm{d}x$ at $x=\bar{x}_2$, we get

$$h_0=(\pi/\sqrt{2})p\cos(p\bar{x}_2-\tfrac{1}{4}\pi). \tag{22}$$

Defining the bifurcation points for this state, we obtain that $\bar{x}_{2c}=\pi/4p$. Substituting $\bar{x}_{2c}$ into (22) gives the bifurcation curve

$$h_c=(\pi/\sqrt{2})(1-\gamma)^{-1/2} \tag{23}$$

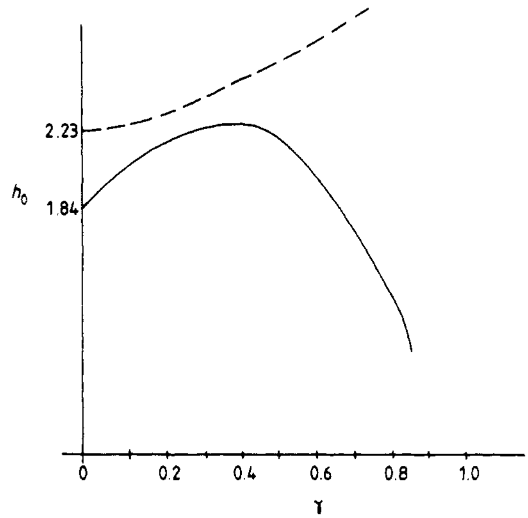

Figure 2.

depicted in figure 2 by a broken curve. The bifurcation points can be also found via the relations (16):

$$F_1(x)=\begin{cases}-s\tan(spx+\theta) & 0<x<\bar{x}_2\\ q\tanh(qpx) & \bar{x}_2<x.\end{cases} \tag{24}$$

Equating the solutions at $x=\bar{x}_2$ gives

$$\tan(ps\bar{x}_2)=qs^{-1}. \tag{25}$$

At the bifurcation point $\omega_0^2=0$ (i.e. $q=s=1$) and from (25) we obtain a value for $\bar{x}_{2c}$ which coincides with the one defined above. From (23) and (25) we obtain the relationship between the eigenfrequency ω^2 and the intensity of the external magnetic field:

$$\omega_0^2\simeq 4-(8/\pi)\sin^{-1}(h_0/h_c). \tag{26}$$

Calculations based on the formula (26) show that this static state is stable in a narrow interval of values of the magnetic field intensity h_0. Taking account of the weak coupling γ leads to magnification of the critical field h_c (for example, ω^2 is non-negative for $2.07<h_0<2.33$ when $\gamma=0$ and for $2.9<h_0<3.14$ when $\gamma=0.5$).

4. The system of junctions with one micro-inhomogeneity

Let us consider semi-infinite JJs in the case when there is an inhomogeneity in one of the junctions. By inhomogeneity we shall mean here a region of local increase in junction resistance (the practical realisation of this situation is discussed in [16]). Approximating an inhomogeneity by a δ function, we rewrite the first equation of the system (1) in the form

$$[1-\mu_1\delta(x-x_1)]\sin\phi_1(x,t)-[\phi_1''(x,t)+\gamma\phi_2''(x,t)]+\alpha_1\dot{\phi}_1(x,t)+\ddot{\phi}_1(x,t)=0 \tag{27}$$

while the second equation will remain unchanged. Here μ_1 characterises the strength of the inhomogeneity and x_1 is its coordinate. Picking out the static states, we come to the following equations:

$$\begin{aligned} &\phi_1''(x)+\gamma\phi_2''(x)=[1-\mu_1\delta(x-x_1)]\sin\phi_1(x) \\ &\phi_2''(x)+\gamma\phi_1''(x)=\sin\phi_2(x) \end{aligned} \tag{28a}$$

$$\begin{aligned} &-[\psi_1''(x)+\gamma\psi_2''(x)]+[1-\mu_1\delta(x-x_1)]\psi_1(x)\cos\phi_1(x)=\omega^2\psi_1(x) \\ &-[\psi_2''(x)+\gamma\psi_1''(x)]+\psi_2(x)\cos\phi_2(x)=\omega^2\psi_2(x). \end{aligned} \tag{28b}$$

On the homogeneous intervals of x the solutions $\phi_1(x)$, $\phi_2(x)$ and $\psi_1(x)$, $\psi_2(x)$ of the system (28) coincide with the corresponding expressions, obtained from (4) and (5) for the homogeneous junctions. At the point x_1, where the inhomogeneity is located, we have the jump condition for the first derivatives

$$\begin{aligned} &\phi_1'(x_1+0)-\phi_1'(x_1-0)=-\mu_1\sin\phi_1(x_1) \\ &\psi_1'(x_1+0)-\psi_1'(x_1-0)=-\mu_1\psi_1(x)\cos\phi_1(x_1) \end{aligned} \tag{29}$$

and continuity of the functions themselves.

Thus, the expressions (11)-(16) together with the conditions (29) now define the states of fluxons in JJS with one micro-inhomogeneity and eigenfrequencies of these states.

Let us consider for definiteness the states in the semi-infinite JJS with the boundary conditions (18), for which $\phi_1(x=0)$, $\phi_1(x=x_1)\in I_1$, while N_2 is an odd number. Writing solutions on homogeneous intervals in accordance with (13) and using the jump condition (29) at the point of inhomogeneity x_1, we eliminate all unknown parameters, except $\bar{x}_2$, and obtain the following equation for $\bar{x}_2$:

$$h_0=\frac{\pi}{2}\,\frac{[C(x_1)\cos p(x_1-\bar{x}_2)+D(x_1)\sin p(x_1-\bar{x}_2)]\cos\tau\bar{x}_2}{A(x_1)\sin\tau\bar{x}_2+B(x_1)\cos\tau\bar{x}_2+(1-p^{-1}C(x_1))\sin p(x_1-\bar{x}_2)-\cos p(x_1-\bar{x}_2)}\,. \tag{30}$$

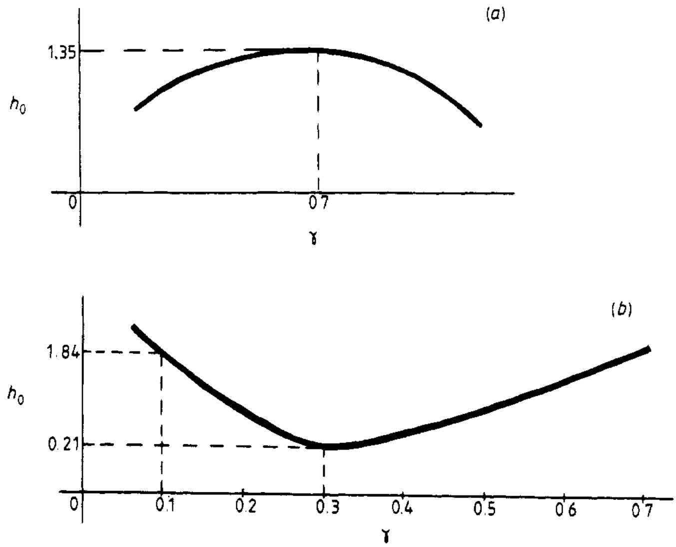

Figure 3.

where

$A(x_1) = \tau^{-1}(\mu_1 + 1 - p \tan px_1)^{-1} \cos \tau x_1$

$B(x_1) = \tau^{-1}[2(p \tan px_1 + \tau \tan \tau x_1) - \mu_1] \sin \tau x_1 + 2(1 - \tau^{-1} \tan \tau x_1) \cos \tau x_1$

$C(x_1) = p(1 + \tan px_1) - \mu_1 \qquad D(x_1) = C(x_1) - 2p.$

If we put $x_1 = 0$ and $\mu = 0$, i.e. consider a homogeneous junction, then (30) coincides with (20).

Defining the bifurcation points from the condition $dh_0/d\bar{x}_2 = 0$, we find the critical values $\bar{x}_{2c}$ and obtain the bifurcation curve $h_0(\gamma)$ (see figure 3). We note that when $\mu_1 \sim 0.1$ the obtained dependence (figure 3a) coincides with the analogous one for the system of homogeneous junctions. In the case $\mu_1 \sim 1$, a richer picture of fluxon states is realised than in the homogeneous case. We propose to carry out a more detailed investigation of this problem in a separate publication.

5. Conclusion

In the framework of the piecewise-linear model we have investigated the problem of propagating fluxons in JJS. The homogeneous semi-infinite JJS in an external magnetic field and some aspects of a JJS with a micro-inhomogeneity in one of the junctions were studied in detail. It was shown that in these systems there exist stable states of fluxons and their number depends on the parameters specifying the problem (magnetic field intensity h_0, the factor $\gamma = \exp(-d/\lambda_L)$, which defines the field attenuation in an S layer of thickness d; the coordinate x_1 and the strength of inhomogeneity μ_1). As was noted, for some critical values of the parameters mentioned a bifurcation of states occurs. Some simple bifurcation surfaces have been presented. Finally, as one of the possible further applications of the approach considered, we would like to mention the problem of fluxons interacting in a system of identical Josephson junctions with different electric currents in them, which is of interest from a practical point of view.

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